EXTENSION OF THE ν-METRIC FOR STABILIZABLE PLANTS OVER H^∞

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Abstract. An abstract ν-metric was introduced in [1], with a view towards extending the classical ν-metric of Vinnicombe from the case of rational transfer functions to more general nonrational transfer function classes of infinite-dimensional linear control systems. Here we give an important concrete special instance of the abstract ν-metric, namely the case when the ring of stable transfer functions is the Hardy algebra H^∞, by verifying that all the assumptions demanded in the abstract set-up are satisfied. This settles the open question implicit in [2].

1. Introduction

We recall the general *stabilization problem* in control theory. Suppose that R is a commutative integral domain with identity (thought of as the class of stable transfer functions) and let $\mathbb{F}(R)$ denote the field of fractions of R. The stabilization problem is: Given $P \in (\mathbb{F}(R))^{p\times m}$ (an unstable plant transfer function), find $C \in (\mathbb{F}(R))^{m\times p}$ (a stabilizing controller transfer function), such that

$$H(P,C) := \begin{bmatrix} P \\ I \end{bmatrix} (I-CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix} \in R^{(p+m)\times(p+m)} \text{ (is stable).}$$

In the *robust stabilization problem*, one goes a step further. One knows that the plant is just an approximation of reality, and so one would really like the controller C to not only stabilize the *nominal* plant P_0, but also all sufficiently close plants P to P_0. The question of what one means by "closeness" of plants thus arises naturally. So one needs a function d defined on pairs of stabilizable plants such that

(1) d is a metric on the set of all stabilizable plants,
(2) d is amenable to computation, and
(3) stabilizability is a robust property of the plant with respect to this metric (that is, whenever a plant P_0 is stabilized by a controller C, then there is a small enough neighbourhood of the plant P_0 consisting of plants which are stabilized by the same controller C).

Such a desirable metric, was introduced by Glenn Vinnicombe in [13] and is called the *ν-metric*. In that paper, essentially R was taken to be the rational functions without poles in the closed unit disk or, more generally, the disk

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algebra, and the most important results were that the ν-metric is indeed a metric on the set of stabilizable plants, and moreover, one has the inequality that if $P_0, P \in \mathbb{S}(R,p,m)$, then

$$\mu_{P,C} \geq \mu_{P_0,C} - d_\nu(P_0, P),$$

where $\mu_{P,C}$ denotes the *stability margin* of the pair (P, C), defined by

$$\mu_{P,C} := \|H(P,C)\|_\infty^{-1}.$$

This implies in particular that stabilizability is a robust property of the plant.

The problem of what happens when R is some other ring of stable transfer functions of infinite-dimensional systems was left open in [13]. This problem of extending the ν-metric from the rational case to transfer function classes of infinite-dimensional systems was addressed in [1]. There the starting point in the approach was abstract. It was assumed that R is any commutative integral domain with identity which is a subset of a Banach algebra S satisfying certain assumptions, labeled (A1)-(A4), which are recalled in Section 2. Then an "abstract" ν-metric was defined in this setup, and it was shown in [1] that it does define a metric on the class of all stabilizable plants. It was also shown there that stabilizability is a robust property of the plant.

In [13], it was suggested that the ν-metric in the case when $R = H^\infty$ might be defined as follows. (Here H^∞ denotes the algebra of bounded and holomorphic functions in the unit disk $\{z \in \mathbb{C} : |z| < 1\}$.) Let P_1, P_2 be unstable plants with the normalized left/right coprime factorizations

$$\begin{aligned} P_1 &= N_1 D_1^{-1} = \widetilde{D}_1^{-1} \widetilde{N}_1, \\ P_2 &= N_2 D_2^{-1} = \widetilde{D}_2^{-1} \widetilde{N}_2, \end{aligned}$$

where $N_1, D_1, N_2, D_2, \widetilde{N}_1, \widetilde{D}_1, \widetilde{N}_2, \widetilde{D}_2$ are matrices with H^∞ entries. Then

$$d_\nu(P_1, P_2) = \begin{cases} \|\widetilde{G}_2 G_1\|_\infty & \text{if } T_{G_1^* G_2} \text{ is Fredholm with Fredholm index 0,} \\ 1 & \text{otherwise.} \end{cases}$$

Here $G_k, \widetilde{G}_k$ arise from P_k ($k = 1, 2$) according to the notational conventions given in Subsection 2.5 below, and $\cdot^*$ has the usual meaning, namely: $G_1^*(\zeta)$ is the transpose of the matrix whose entries are complex conjugates of the entries of the matrix $G_1(\zeta)$, for $\zeta \in \mathbb{T}$. Also in the above, for a matrix $M \in (L^\infty)^{p \times m}$, T_M denotes the *Toeplitz operator* from $(H^2)^m$ to $(H^2)^p$, given by

$$T_M \varphi = \mathrm{P}_{(H^2)^p}(M\varphi) \quad (\varphi \in (H^2)^m)$$

where $M\varphi$ is considered as an element of $(L^2)^p$ and $\mathrm{P}_{(H^2)^p}$ denotes the canonical orthogonal projection from $(L^2)^p$ onto $(H^2)^p$.

In [2], we showed that the above does work for the case when R is the smaller class QA of quasianalytic functions in the unit disk. We proved this by showing that this case is just a special instance of the abstract ν-metric

introduced in [1]. A perusal of the extensive literature on Fredholm theory of Toeplitz operators from the 1970s lead to this choice of $R = QA$ and $S = QC$ (the class of quasicontinuous functions) as conceivably the most general subalgebras of H^∞ and L^∞ which fit the setup of [1].

In this article, we use a different idea to tackle the problem of defining a new metric in the case when $R = H^\infty$. We first notice that when R is the disk algebra $A(\mathbb{D})$, then there is no problem in defining the ν-metric; see [1, §5.1]. We then handle the H^∞ case by using the observation that the restrictions of a function $f \in H^\infty$ to the smaller disks with radii $r < 1$ give rise to elements in the disk algebra by dilating these restrictions to bigger disks of radius 1. In other words, f_r defined via

$$f_r(z) = f(rz) \quad (z \in \mathbb{D}).$$

are all elements of $A(\mathbb{D})$. We then use these restrictions in a suitable manner to define the ν-metric.

The paper is organized as follows:

(1) In Section 2, we recall the general setup with the assumptions and the abstract metric d_ν from [1].
(2) In Section 3, we specialize R to a concrete ring of stable transfer functions, namely $R = H^\infty$, and show that our abstract assumptions hold in this particular case. Moreover in the Subsection 3.2, we will show that when our extended ν-metric is restricted to rational plants, we obtain the classical ν-metric, hence showing that we have obtained a genuine extension.

2. Recap of the abstract ν-metric

We recall the setup from [1]:

(A1) R is commutative integral domain with identity.
(A2) S is a unital commutative complex semisimple Banach algebra with an involution $\cdot^*$, such that $R \subset S$. We use inv S to denote the invertible elements of S.
(A3) There exists a map $\iota : \text{inv } S \to G$, where $(G, +)$ is an Abelian group with identity denoted by $\circ$, and ι satisfies
  (I1) $\iota(ab) = \iota(a) + \iota(b)$ $(a, b \in \text{inv } S)$.
  (I2) $\iota(a^*) = -\iota(a)$ $(a \in \text{inv } S)$.
  (I3) ι is locally constant, that is, ι is continuous when G is equipped with the discrete topology.
(A4) $x \in R \cap (\text{inv } S)$ is invertible as an element of R if and only if $\iota(x) = \circ$.

We recall the following standard definitions from the factorization approach to control theory.

2.1. **The notation $\mathbb{F}(R)$:** $\mathbb{F}(R)$ denotes the field of fractions of R.

2.2. **The notation F^*:** If $F \in R^{p\times m}$, then $F^* \in S^{m\times p}$ is the matrix with the entry in the ith row and jth column given by F^*_{ji}, for all $1 \leq i \leq p$, and all $1 \leq j \leq m$.

2.3. **Right coprime/normalized coprime factorization:** For a matrix $P \in (\mathbb{F}(R))^{p\times m}$, a factorization $P = ND^{-1}$, where N, D are matrices with entries from R, is called a *right coprime factorization of* P if there exist matrices X, Y with entries from R such that $XN + YD = I_m$. If moreover $N^*N + D^*D = I_m$, then the right coprime factorization is referred to as a *normalized* right coprime factorization of P.

2.4. **Left coprime/normalized coprime factorization:** For a matrix $P \in (\mathbb{F}(R))^{p\times m}$, a factorization $P = \widetilde{D}^{-1}\widetilde{N}$, where $\widetilde{N}, \widetilde{D}$ are matrices with entries from R, is called a *left coprime factorization of* P if there exist matrices $\widetilde{X}, \widetilde{Y}$ with entries from R such that $\widetilde{N}\widetilde{X} + \widetilde{D}\widetilde{Y} = I_p$. If moreover $\widetilde{N}\widetilde{N}^* + \widetilde{D}\widetilde{D}^* = I_p$, then the left coprime factorization is referred to as a *normalized* left coprime factorization of P.

2.5. **The notation $G, \widetilde{G}, K, \widetilde{K}$:** Given $P \in (\mathbb{F}(R))^{p\times m}$ with normalized right and left factorizations $P = ND^{-1}$ and $P = \widetilde{D}^{-1}\widetilde{N}$, respectively, we introduce the following matrices with entries from R:

$$G = \begin{bmatrix} N \\ D \end{bmatrix} \quad \text{and} \quad \widetilde{G} = \begin{bmatrix} -\widetilde{D} & \widetilde{N} \end{bmatrix}.$$

Similarly, given a $C \in (\mathbb{F}(R))^{m\times p}$ with normalized right and left factorizations $C = N_C D_C^{-1}$ and $C = \widetilde{D}_C^{-1}\widetilde{N}_C$, respectively, we introduce the following matrices with entries from R:

$$K = \begin{bmatrix} D_C \\ N_C \end{bmatrix} \quad \text{and} \quad \widetilde{K} = \begin{bmatrix} -\widetilde{N}_C & \widetilde{D}_C \end{bmatrix}.$$

2.6. **The notation $\mathbb{S}(R, p, m)$:** $\mathbb{S}(R, p, m)$ denotes the set of all elements $P \in (\mathbb{F}(R))^{p\times m}$ that possess a normalized right coprime factorization and a normalized left coprime factorization.

We now recall the definition of the metric d_ν on $\mathbb{S}(R, p, m)$. But first we specify the norm we use for matrices with entries from S.

Definition 2.1 ($\|\cdot\|_{S,\infty}$)**.** Let $\mathfrak{M}$ denote the maximal ideal space of the Banach algebra S. For a matrix $M \in S^{p\times m}$, we set

$$\|M\|_{S,\infty} = \max_{\varphi\in\mathfrak{M}} \|\mathbf{M}(\varphi)\|, \tag{2.1}$$

and refer to it as the *Gelfand norm*. Here $\mathbf{M}$ denotes the entry-wise Gelfand transform of M, and $\|\cdot\|$ denotes the induced operator norm from $\mathbb{C}^m$ to $\mathbb{C}^p$. For the sake of concreteness, we fix the standard Euclidean norms on the vector spaces $\mathbb{C}^m$ to $\mathbb{C}^p$.

The maximum in (2.1) exists since $\mathfrak{M}$ is a compact space when it is equipped with Gelfand topology, that is, the weak-$*$ topology induced from $\mathcal{L}(S;\mathbb{C})$. Since we have assumed S to be semisimple, the Gelfand transform

$$\widehat{\cdot} : S \to \widehat{S}\ (\subset C(\mathfrak{M},\mathbb{C}))$$

is an injective map. If $M \in S^{1\times 1} = S$, then we note that there are two norms available for M: the one as we have defined above, namely $\|M\|_{S,\infty}$, and the norm $\|M\|_S$ of M as an element of the Banach algebra S. But throughout this article, we will use the norm given by (2.1).

Definition 2.2 (Abstract ν-metric d_ν)**.** For $P_1, P_2 \in \mathbb{S}(R,p,m)$, with the normalized left/right coprime factorizations

$$\begin{aligned} P_1 &= N_1 D_1^{-1} = \widetilde{D}_1^{-1}\widetilde{N}_1, \\ P_2 &= N_2 D_2^{-1} = \widetilde{D}_2^{-1}\widetilde{N}_2, \end{aligned}$$

we define

$$d_\nu(P_1,P_2) := \left\{ \begin{array}{ll} \|\widetilde{G}_2 G_1\|_{S,\infty} & \text{if } \det(G_1^* G_2) \in \operatorname{inv} S \text{ and} \\ & \quad\ \iota(\det(G_1^* G_2)) = \circ, \\ 1 & \text{otherwise,} \end{array} \right. \tag{2.2}$$

where the notation is as in Subsections 2.1-2.6.

The following was proved in [1]:

Theorem 2.3. *d_ν given by* (2.2) *is a metric on $\mathbb{S}(R,p,m)$.*

Definition 2.4. Given $P \in (\mathbb{F}(R))^{p\times m}$ and $C \in (\mathbb{F}(R))^{m\times p}$, the *stability margin* of the pair (P,C) is defined by

$$\mu_{P,C} = \left\{ \begin{array}{ll} \|H(P,C)\|_{S,\infty}^{-1} & \text{if } P \text{ is stabilized by } C, \\ 0 & \text{otherwise.} \end{array} \right.$$

The number $\mu_{P,C}$ can be interpreted as a measure of the performance of the closed loop system comprising P and C: larger values of $\mu_{P,C}$ correspond to better performance, with $\mu_{P,C} > 0$ if and only if C stabilizes P.

The following was proved in [1]:

Theorem 2.5. *If $P_0, P \in \mathbb{S}(R,p,m)$ and $C \in \mathbb{S}(R,m,p)$, then*

$$\mu_{P,C} \geq \mu_{P_0,C} - d_\nu(P_0,P).$$

The above result says that stabilizability is a robust property of the plant, since if C stabilizes P_0 with a stability margin $\mu_{P,C} > m$, and P is another plant which is close to P_0 in the sense that $d_\nu(P,P_0) \leq m$, then C is also guaranteed to stabilize P.

3. The ν-metric when $R = H^\infty$

Let H^∞ be the Hardy algebra, consisting of all bounded and holomorphic functions defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

We will now introduce a Banach algebra, $C_{\rm b}(\mathbb{A}_\rho)$, which will serve as the Banach algebra S in our abstract set up.

Notation 3.1. Given $\rho \in (0,1)$, let $\mathbb{A}_\rho$ be the open annulus

$$\mathbb{A}_\rho := \{z \in \mathbb{C} : \rho < |z| < 1\}.$$

We set $C_{\rm b}(\mathbb{A}_\rho) = \{F : \mathbb{A}_\rho \to \mathbb{C} : f \text{ is continuous and bounded on } \mathbb{A}_\rho\}$.

Proposition 3.2. *Let $\rho \in (0,1)$. With the norm defined by*

$$\|F\|_\infty := \sup_{z \in \mathbb{A}_\rho} |F(z)| \textit{ for } F \in C_{\rm b}(\mathbb{A}_\rho),$$

$C_{\rm b}(\mathbb{A}_\rho)$ is a unital semisimple complex Banach algebra with the involution $\cdot^*$ defined by

$$(F^*)(z) = \overline{F(z)} \quad (z \in \mathbb{A}_\rho,\ F \in C_{\rm b}(\mathbb{A}_\rho)).$$

Proof. The verification of the claims is straightforward. We just give the proof of the semisimplicity. Recall that a commutative complex Banach algebra is called semisimple if its radical ideal, namely the intersection of all the maximal ideals of the Banach algebra is 0. We also know that kernels of complex homomorphisms are maximal ideals. For $z \in \mathbb{A}_\rho$, the map $\varphi_z : C_{\rm b}(\mathbb{A}_\rho) \to \mathbb{C}$, given by $\varphi_z(F) = F(z)$ for $F \in C_{\rm b}(\mathbb{A}_\rho)$, is a complex homomorphism. We have

$$\bigcap_{z \in \mathbb{A}_\rho} \ker \varphi_z = \{0\}.$$

Since the radical ideal is contained in the intersection of the kernels of the complex homomorphisms φ_z, $z \in \mathbb{A}_\rho$, it must be zero. $\square$

Proposition 3.3. *Let $\rho \in (0,1)$. For $f \in H^\infty$, define $\mathcal{I} : H^\infty \to C_{\rm b}(\mathbb{A}_\rho)$ by*

$$(\mathcal{I}(f))(z) = f(z) \quad (z \in \mathbb{A}_\rho,\ f \in H^\infty).$$

Then $\mathcal{I}$ is an injective map.

Proof. The map $\mathcal{I}$ is a linear transformation. Suppose that $\mathcal{I}(f) = 0$ for some $f \in H^\infty$. This means that the restriction of f to the annulus $\mathbb{A}_\rho$ is identically 0, and as f is holomorphic in $\mathbb{D}$, f must be zero in the whole disk $\mathbb{D}$. Hence $f = 0$. $\square$

Henceforth we will identify H^∞ as a subset of $C_{\rm b}(\mathbb{A}_\rho)$ via this map $\mathcal{I}$.

We will now define an index on invertible elements of $S = C_{\rm b}(\mathbb{A}_\rho)$.

Notation 3.4. We use the notation $C(\mathbb{T})$ for the Banach algebra of complex-valued continuous functions defined on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, with all operations defined pointwise, with the supremum norm:

$$\|f\|_\infty = \sup_{\zeta \in \mathbb{T}} |f(\zeta)| \text{ for } f \in C(\mathbb{T}),$$

and the involution $\cdot^*$ defined pointwise:

$$f^*(\zeta) = \overline{f(\zeta)} \quad (\zeta \in \mathbb{T}).$$

If $F \in \text{inv } C_{\text{b}}(\mathbb{A}_\rho)$, then for each $r \in (\rho, 1)$, the map $F_r : \mathbb{T} \to \mathbb{C}$, given by

$$F_r(\zeta) = F(r\zeta) \quad (\zeta \in \mathbb{T}),$$

belongs to inv $C(\mathbb{T})$, and so each F_r has a well-defined (integral) winding number $w(F_r) \in \mathbb{Z}$ with respect to 0.

Moreover, we now show by the local constancy of the winding number $w : \text{inv } C(\mathbb{T}) \to \mathbb{Z}$, that $r \mapsto w(F_r)$ is constant on $(\rho, 1)$.

Proposition 3.5. *If $F \in \text{inv } C_{\text{b}}(\mathbb{A}_\rho)$, and $\rho < r < r' < 1$, then*

$$w(F_r) = w(F_r').$$

Proof. We use the fact that the winding numbers $w(\varphi)$, $w(\psi)$ with respect to 0 of $\varphi, \psi : \mathbb{T} \to \mathbb{C} \setminus \{0\}$, are the same if φ, ψ are homotopic; see for example [3, §2.7.10, p.50].

As the annulus $K := \{z \in \mathbb{C} : r \leq |z| \leq r'\}$ is compact, it follows that there is a $m > 0$ such that $F(z)$ lies in $\mathbb{C} \setminus m\mathbb{D}$ for all $z \in K$. Also, F is uniformly continuous on K, and so we can choose N large enough so that with

$$r_n := r + (r' - r) \cdot \frac{n}{N}, \quad n = 0, 1, \dots, N,$$

we have that

$$\|F_{r_n} - F_{r_{n+1}}\|_\infty < \frac{m}{2}, \quad n = 0, 1, 2, \dots, N-1.$$

Fix an $n \in \{0, 1, 2, \dots, N-1\}$. Set $\varphi = F_{r_n}$ and $\psi = F_{r_{n+1}}$. Then φ, ψ belong to inv $C(\mathbb{T})$. Consider the map $H : \mathbb{T} \times [0, 1] \to \mathbb{C} \setminus \{0\}$ defined by $H(\zeta, t) = \varphi(\zeta) + t(\psi(\zeta) - \varphi(\zeta))$, $\zeta \in \mathbb{T}$, $t \in [0, 1]$. Since

$$|\varphi(\zeta) + t(\psi(\zeta) - \varphi(\zeta))| \geq |\varphi(\zeta)| - |t(\psi(\zeta) - \varphi(\zeta))| \geq m - 1 \cdot (m/2) = m/2 > 0,$$

H is well-defined. H is a homotopy from φ to ψ. In particular it follows from the above that $\psi = H(\cdot, 1) \in \text{inv } C(\mathbb{T})$, and that the winding numbers of φ and ψ are identical. So it follows that

$$w(F_r) = w(F_{r_0}) = w(F_{r_1}) = \cdots = w(F_{r_N}) = w(F_r').$$

This completes the proof. □

Notation 3.6. We now define the map $W : \text{inv } C_{\text{b}}(\mathbb{A}_\rho) \to \mathbb{Z}$ by setting

$$W(F) = w(F_r) \quad (r \in (\rho, 1),\ F \in \text{inv } C_{\text{b}}(\mathbb{A}_\rho)).$$

By the preceding discussion, we see that W is well-defined.

We will now prove a sequence of results aimed towards verifying the assumptions (A3) and (A4) in our abstract setup.

Proposition 3.7. *Let $\rho \in (0, 1)$. If $F, G \in \text{inv } C_{\text{b}}(\mathbb{A}_\rho)$, then*

$$W(FG) = W(F) + W(G).$$

Proof. For $f, g \in \text{inv } C(\mathbb{T})$, we have $w(fg) = w(f) + w(g)$, and so it follows that for $F, G \in \text{inv } C_{\text{b}}(\mathbb{A}_\rho)$, and $r \in (\rho, 1)$,

$$W(FG) = w((FG)_r) = w(F_r \cdot G_r) = w(F_r) + w(G_r) = W(F) + W(G).$$

This completes the proof. $\square$

Proposition 3.8. *Let $\rho \in (0,1)$. If $F \in \text{inv } C_{\text{b}}(\mathbb{A}_\rho)$, then*

$$W(F^*) = -W(F).$$

Proof. For $f \in \text{inv } C(\mathbb{T})$, $w(\overline{f(\cdot)}) = -w(f)$. So if $F \in \text{inv } C_{\text{b}}(\mathbb{A}_\rho)$,

$$W(F^*) = w((F^*)_r) = w((F_r)^*) = -w(F_r) = -W(F).$$

This completes the proof. $\square$

Proposition 3.9. *Let $\rho \in (0,1)$. Then $W : \text{inv } C_{\text{b}}(\mathbb{A}_\rho) \to \mathbb{Z}$ is locally constant, that is, it is continuous when $\mathbb{Z}$ is equipped with the discrete topology.*

Proof. Let $F \in \text{inv } C_{\text{b}}(\mathbb{A}_\rho)$. Let $r \in (\rho, 1)$. By the local constancy of the map $w : \text{inv } C(\mathbb{T}) \to \mathbb{Z}$, it follows that there is a $\delta > 0$ such that for all $h \in \text{inv } C(\mathbb{T})$ satisfying $\|F_r - h\|_\infty < \delta$, we have $w(F_r) = w(h)$. Hence we have $W(F) = w(F_r) = w(H_r) = W(H)$ for all $H \in \text{inv } C_{\text{b}}(\mathbb{A}_\rho)$ satisfying $\|F - H\|_\infty < \delta$. This proves the desired local constancy of W. $\square$

Finally we have the following analogue of the classical Nyquist criterion.

Proposition 3.10. *Let $\rho \in (0,1)$. Suppose that $f \in H^\infty$ is such that $\mathcal{I}(f) \in \text{inv } C_{\text{b}}(\mathbb{A}_\rho)$. Then f is invertible as an element of H^∞ if and only if $W(\mathcal{I}(f)) = 0$.*

Proof. ("If" part) Let $g \in H^\infty$ be the inverse of f. For each $r \in (\rho, 1)$, $f_r \in A(\mathbb{D})$ defined by $f_r(z) = f(rz)$ $(z \in \mathbb{D})$, is invertible in $A(\mathbb{D})$. Then $(\mathcal{I}(f))_r = f_r$. By the Nyquist criterion for $A(\mathbb{D})$, $\varphi \in A(\mathbb{D}) \bigcap \text{inv } C(\mathbb{T})$ is invertible in $A(\mathbb{D})$ if and only if $w(\varphi) = 0$ [1, Lemma 5.2]. Thus $w(f_r) = 0$. Hence $W(\mathcal{I}(f)) = w((\mathcal{I}(f))_r) = w(f_r) = 0$, completing the proof of the "if" part.

("Only if" part) Let $G \in C_{\text{b}}(\mathbb{A}_\rho)$ be the inverse of $F := \mathcal{I}(f)$. If $r \in (\rho, 1)$, then $f_r := f(r\cdot) \in A(\mathbb{D})$ and $f_r \in \text{inv } C(\mathbb{T})$. Since $W(F) = w(f_r) = 0$, it follows again by the Nyquist criterion for the disk algebra recalled above, that f_r is invertible in $A(\mathbb{D})$. In other words, $f(rz) \neq 0$ for all $z \in \mathbb{D}$. It follows from here, as the choice of $r \in (\rho, 1)$ was arbitrary, that $f(z) \neq 0$ for all $z \in \mathbb{D}$, that is, f has a pointwise inverse $g : \mathbb{D} \to \mathbb{C}$. Moreover, g is holomorphic in $\mathbb{D}$. We have $f(z)g(z) = f(z)G(z) = 1$ $(\rho < |z| < 1)$, and so it follows that $G(z) = g(z)$ $(\rho < |z| < 1)$. Hence by the maximum modulus principle,

$$\sup_{z \in \mathbb{D}} |g(z)| = \sup_{1 > |z| > \rho} |g(z)| \leq \|G\|_\infty < +\infty,$$

showing that $g \in H^\infty$. Consequently, $f \in \text{inv } H^\infty$. This completes the proof of the "only if" part. $\square$

Theorem 3.11. *Let $\rho \in (0,1)$. Set*

$$\begin{array}{rcl} R & := & H^\infty, \\ S & := & C_{\rm b}(\mathbb{A}_\rho), \\ G & := & \mathbb{Z}, \\ \iota & := & W. \end{array}$$

Then (A1)-(A4) *are satisfied.*

Proof. Since H^∞ is a commutative integral domain with identity, (A1) holds.

(A2) follows from the results in Propositions 3.2 and 3.3. Indeed, the set $C_{\rm b}(\mathbb{A}_\rho)$ is a unital, commutative, complex, semisimple Banach algebra with the involution $\cdot^*$ defined earlier in Proposition 3.2. Moreover, the map $\mathcal{I} : H^\infty \to C_{\rm b}(\mathbb{A}_\rho)$ is injective.

The map $W : \text{inv } C_{\rm b}(\mathbb{A}_\rho) \to \mathbb{Z}$ satisfies (I1), (I2), (I3) by Propositions 3.7, 3.8, 3.9. Thus (A3) holds.

Finally (A4) has been verified in Proposition 3.10. □

The definition of the abstract ν-metric given in Definition 2.2, now takes the following concrete form. For $P_1, P_2 \in \mathbb{S}(H^\infty, p, m)$, with the normalized left/right coprime factorizations

$$\begin{array}{rcl} P_1 & = & N_1 D_1^{-1} = \widetilde{D}_1^{-1} \widetilde{N}_1, \\ P_2 & = & N_2 D_2^{-1} = \widetilde{D}_2^{-1} \widetilde{N}_2, \end{array}$$

we define

$$d_\nu(P_1, P_2) = \left\{ \begin{array}{ll} \|\widetilde{G}_2 G_1\|_{C_{\rm b}(\mathbb{A}_\rho),\infty} & \text{if } \det(G_1^* G_2) \in \text{inv } C_{\rm b}(\mathbb{A}_\rho) \text{ and} \\ & \quad W(\det(G_1^* G_2)) = 0, \\ 1 & \text{otherwise,} \end{array} \right. \tag{3.1}$$

where the notation is as in Subsections 2.1-2.6.

We will now show that in fact the Gelfand norm $\|\cdot\|_{C_{\rm b}(\mathbb{A}_\rho),\infty}$ above can be replaced by the usual $\|\cdot\|_\infty$ norm for elements from H^∞.

Lemma 3.12. *Let $\rho \in (0,1)$. Let $A \in (H^\infty)^{p\times m}$. Then*

$$\|A\|_{C_{\rm b}(\mathbb{A}_\rho),\infty} = \|A\|_\infty := \sup_{z\in\mathbb{D}} \boldsymbol{|} A(z) \boldsymbol{|}.$$

Proof. We first note that $C_{\rm b}(\mathbb{A}_\rho)$ is a C^*-algebra. Indeed, for $F \in C_{\rm b}(\mathbb{A}_\rho)$,

$$\|F^* F\|_\infty = \sup_{z\in\mathbb{A}_\rho} |\overline{F(z)} F(z)| = \sup_{z\in\mathbb{A}_\rho} |F(z)|^2 = \|F\|_\infty^2.$$

Therefore (by the Gelfand-Naimark Theorem; see [10, Theorem 11.18]) for all $F \in C_{\rm b}(\mathbb{A}_\rho)$, we have

$$\|F\|_\infty = \max_{\varphi \in \mathfrak{M}(C_{\rm b}(\mathbb{A}_\rho))} |\widehat{F}(\varphi)| =: \|F\|_{C_{\rm b}(\mathbb{A}_\rho),\infty}.$$

In the sequel, we use the notation $\sigma_{\max}(X)$ (for $X \in \mathbb{C}^{p\times m}$) to mean the largest singular value of X, that is, the square root of the largest eigenvalue of X^*X or XX^*. The map $\sigma_{\max}(\cdot) : \mathbb{C}^{p\times m} \to [0, \infty)$ is continuous.

Now let $F \in (C_{\mathrm{b}}(\mathbb{A}_\rho))^{p\times m}$. Then $\sigma_{\max}(\widehat{F}(\cdot))$ is a continuous function on the maximal ideal space $\mathfrak{M}(C_{\mathrm{b}}(\mathbb{A}_\rho))$, and so (by [10, Theorem 11.18, p.289]) there exists an element $\mu_1 \in C_{\mathrm{b}}(\mathbb{A}_\rho)$ such that

$$\widehat{\mu_1}(\varphi) = \sigma_{\max}(\widehat{F}(\varphi)) \text{ for all } \varphi \in \mathfrak{M}(C_{\mathrm{b}}(\mathbb{A}_\rho)).$$

Also, the map $z \mapsto \sigma_{\max}(F(z))$ is continuous on $\mathbb{A}_\rho$. Moreover, we have that

$$\sup_{z\in\mathbb{A}_\rho} \sigma_{\max}(F(z)) = \sup_{z\in\mathbb{A}_\rho} \| F(z) \| < \infty.$$

Consequently, if we define $\mu_2(z) := \sigma_{\max}(F(z))$ $(z \in \mathbb{A}_\rho)$, then $\mu_2 \in C_{\mathrm{b}}(\mathbb{A}_\rho)$. This μ_2 satisfies the equation $\det(\mu_2^2 I - A^*A) = 0$, which yields, by taking Gelfand transforms, that $\det((\widehat{\mu_2}(\varphi))^2 I - (\widehat{A}(\varphi))^*\widehat{A}(\varphi)) = 0$ for all φ belonging to $\mathfrak{M}(C_{\mathrm{b}}(\mathbb{A}_\rho))$. Hence there holds

$$|\widehat{\mu_2}(\varphi)| \le \sigma_{\max}(\widehat{A}(\varphi)) = \widehat{\mu_1}(\varphi) \text{ for all } \varphi \in \mathfrak{M}(C_{\mathrm{b}}(\mathbb{A}_\rho)). \tag{3.2}$$

Also, since $\det((\widehat{\mu_1}(\varphi))^2 I - (\widehat{A}(\varphi))^*\widehat{A}(\varphi)) = 0$ for $\varphi \in \mathfrak{M}(C_{\mathrm{b}}(\mathbb{A}_\rho))$, it follows that $\det(\mu_1^2 I - A^*A) = 0$, which gives the inequality

$$|\mu_1(z)| \le \sigma_{\max}(F(z)) = \mu_2(z) \text{ for all } z \in \mathbb{A}_\rho. \tag{3.3}$$

It now follows from (3.2) and (3.3) that $\|\mu_1\|_{C_{\mathrm{b}}(\mathbb{A}_\rho)} = \|\mu_2\|_{C_{\mathrm{b}}(\mathbb{A}_\rho)}$, and so

$$\sup_{z\in\mathbb{A}_\rho} \sigma_{\max}(F(z)) = \max_{\varphi\in\mathfrak{M}(C_{\mathrm{b}}(\mathbb{A}_\rho))} \sigma_{\max}(\widehat{F}(\varphi)).$$

Consequently, $\|F\|_{C_{\mathrm{b}}(\mathbb{A}_\rho),\infty} = \|F\|_\infty := \sup_{z\in\mathbb{A}_\rho} \| F(z) \|$.

Now suppose that $A \in (H^\infty)^{p\times m}$. Then we have

$$\|A\|_{C_{\mathrm{b}}(\mathbb{A}_\rho),\infty} = \sup_{z\in\mathbb{A}_\rho} \| A(z) \| = \sup_{z\in\mathbb{D}} \| A(z) \| = \|A\|_\infty,$$

we we have used the vector valued version of the Maximum Modulus Principle (see for example [9, p.50]) to obtain the second equality. This completes the proof. □

In light of the above result, the abstract ν-metric now takes the following form.

For $P_1, P_2 \in \mathbb{S}(H^\infty, p, m)$, with the normalized left/right coprime factorizations

$$\begin{aligned} P_1 &= N_1 D_1^{-1} = \widetilde{D}_1^{-1}\widetilde{N}_1, \\ P_2 &= N_2 D_2^{-1} = \widetilde{D}_2^{-1}\widetilde{N}_2, \end{aligned}$$

we define

$$d_\nu(P_1, P_2) := \left\{ \begin{array}{ll} \|\widetilde{G}_2 G_1\|_\infty & \text{if } \det(G_1^* G_2) \in \text{inv } C_{\mathrm{b}}(\mathbb{A}_\rho) \text{ and} \\ & \quad W(\det(G_1^* G_2)) = 0, \\ 1 & \text{otherwise,} \end{array} \right. \tag{3.4}$$

where the notation is as in Subsections 2.1-2.6.

Remark 3.13. We also remark that the set $\mathbb{S}(H^\infty, p, m)$ coincides with the set of stabilizable plants, using the following two facts:

(1) A plant is stabilizable over H^∞ if and only if it possesses a coprime factorization. (See [6] and [12].)

(2) A *normalized* coprime factorization over H^∞ exists whenever a coprime factorization exists over H^∞. (See for example [7, Theorem 1.1].)

Summarizing, our main result is the following, where the stability margin of a pair $(P, C) \in \mathbb{S}(H^\infty, p, m) \times \mathbb{S}(H^\infty, m, p)$ is

$$\mu_{P,C} = \begin{cases} \|H(P,C)\|_\infty^{-1} & \text{if } P \text{ is stabilized by } C, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 3.14. *d_ν given by* (3.4) *is a metric on the set of stabilizable plants over H^∞. Moreover, if P_0, P belong to $\mathbb{S}(H^\infty, p, m)$ and $C \in \mathbb{S}(H^\infty, m, p)$, then $\mu_{P,C} \geq \mu_{P_0,C} - d_\nu(P_0, P)$.*

3.1. **Irrelevance of $\rho \in (0,1)$ in the definition of the ν-metric for $\mathbb{S}(H^\infty, p, m)$.** Consider the condition

$$(C): \boxed{\det(G_1^* G_2) \in \text{inv } C_{\text{b}}(\mathbb{A}_\rho) \text{ and } W(\det(G_1^* G_2)) = 0.}$$

Clearly only the tail end of the winding numbers are relevant, and so the noninvertibility in $C_{\text{b}}(\mathbb{A}_\rho)$ owing to the noninveribility of $\det((G_1|_{r\mathbb{T}})^* G_2|_{r\mathbb{T}})$ for small r's in $(\rho, 1)$ should not really matter. We remedy this problem by taking the pointwise limit as $\rho \nearrow 1$ of the ν-metrics corresponding to the ρ's in $(0, 1)$.

For $\rho \in (0,1)$, let d_ν^ρ denote the ν-metric given by (3.4). Define d_ν^∞ on plant pairs from $\mathbb{S}(H^\infty, p, m)$ as follows. For $P_1, P_2 \in \mathbb{S}(H^\infty, p, m)$,

$$d_\nu^\infty(P_1, P_2) := \lim_{\rho \to 1} d_\nu^\rho(P_1, P_2). \tag{3.5}$$

We note that if the condition (C) is satisfied corresponding to ρ for some $\rho \in (0,1)$, then it is also satisfies for all ρ' satisfying $\rho \leq \rho' < 1$. This shows that the numbers $d_\nu^\rho(P_1, P_2)$, $\rho \in (0,1)$, are all equal for all ρ's beyond a certain $\rho_{\text{c}} \in (0,1)$. Thus d_ν^∞, given by (3.5), is well-defined. We will now check that d_ν^∞ is a metric on $\mathbb{S}(H^\infty, p, m)$ and that with this metric, stabilizability is a robust property of plants.

Theorem 3.15. *d_ν^∞ given by* (3.5) *is a metric on the set of stabilizable plants over H^∞. Moreover, if $P_0, P \in \mathbb{S}(H^\infty, p, m)$ and $C \in \mathbb{S}(H^\infty, m, p)$, then $\mu_{P,C} \geq \mu_{P_0,C} - d_\nu^\infty(P_0, P)$.*

Proof. We first show that d_ν^∞ defines a metric on $\mathbb{S}(H^\infty, p, m)$.

(D1) For $P_1, P_2 \in \mathbb{S}(H^\infty, p, m)$, since $d_\nu^\rho(P_1, P_2) \geq 0$ for each $\rho \in (0,1)$,

$$d_\nu^\infty(P_1, P_2) = \lim_{\rho \to 1} d_\nu^\rho(P_1, P_2) \geq 0.$$

For $P \in \mathbb{S}(H^\infty, p, m)$, $d_\nu^\infty(P, P) = \lim\limits_{\rho \to 1} d_\nu^\rho(P, P) = \lim\limits_{\rho \to 1} 0 = 0.$

Finally, if $d_\nu^\infty(P_1,P_2)=0$ for $P_1,P_2\in\mathbb{S}(H^\infty,p,m)$, then since we have seen that the numbers $d_\nu^\rho(P_1,P_2)$, $\rho\in(0,1)$, are all equal for all ρ's close enough to 1, it must be the case that $d_\nu^\rho(P_1,P_2)=0$ for all ρ's close enough to 1, and so $P_1=P_2$.

(D2) If $P_1,P_2\in\mathbb{S}(H^\infty,p,m)$, then we have

$$d_\nu^\infty(P_1,P_2)=\lim_{\rho\to1}d_\nu^\rho(P_1,P_2)=\lim_{\rho\to1}d_\nu^\rho(P_2,P_1)=d_\nu^\infty(P_2,P_1).$$

(D3) Finally, for all $P_1,P_2,P_3\in\mathbb{S}(H^\infty,p,m)$, passing the limit as $\rho\to1$ in the triangle inequalities

$$d_\nu^\rho(P_1,P_3)\le d_\nu^\rho(P_1,P_2)+d_\nu^\rho(P_2,P_3)\quad(\rho\in(0,1)),$$

yields the triangle inequality $d_\nu^\infty(P_1,P_3)\le d_\nu^\infty(P_1,P_2)+d_\nu^\infty(P_2,P_3)$.

Thus d_ν^∞ defines a metric on $\mathbb{S}(H^\infty,p,m)$. Next we show that stabilizability is a robust property of the plant. Let P_0, P belong to $\mathbb{S}(H^\infty,p,m)$ and $C\in\mathbb{S}(H^\infty,m,p)$. Then $\mu_{P,C}\ge\mu_{P_0,C}-d_\nu^\rho(P_0,P)$ $(\rho\in(0,1))$. Again passing the limit as $\rho\to1$, we obtain $\mu_{P,C}\ge\mu_{P_0,C}-d_\nu^\infty(P_0,P)$. This completes the proof. $\square$

3.2. **d_ν^∞ is an extension of the "classical" ν-metric.** In [13], the ν-metric for rational plants (and more generally elements of $\mathbb{S}(A(\mathbb{D}),p,m)$) was defined as follows. For $P_1,P_2\in\mathbb{S}(A(\mathbb{D}),p,m)$, with the normalized left/right coprime factorizations

$$\begin{aligned}P_1&=N_1D_1^{-1}=\widetilde{D}_1^{-1}\widetilde{N}_1,\\ P_2&=N_2D_2^{-1}=\widetilde{D}_2^{-1}\widetilde{N}_2,\end{aligned}$$

we define

$$d_{\nu,\mathrm{classical}}(P_1,P_2):=\begin{cases}\|\widetilde{G}_2G_1\|_\infty & \text{if } \det(G_1^*G_2)\in\operatorname{inv}C(\mathbb{T})\text{ and}\\ & \quad w(\det(G_1^*G_2))=0,\\ 1 & \text{otherwise,}\end{cases}\tag{3.6}$$

where the notation is as in Subsections 2.1-2.6.

In this subsection we will show that our ν-metric, defined by (3.5), coincides exactly with the above metric defined by (3.6), when the data P_1,P_2 belong to $\mathbb{S}(A(\mathbb{D}),p,m)$ (instead of the bigger set $\mathbb{S}(H^\infty,p,m)$).

Theorem 3.16. *Let $P_1,P_2\in\mathbb{S}(A(\mathbb{D}),p,m)$. Then*

$$d_{\nu,\mathrm{classical}}(P_1,P_2)=d_\nu^\infty(P_1,P_2).$$

Proof. Let $d_{\nu,\mathrm{classical}}(P_1,P_2)<1$. Then $\det(G_1^*G_2)\in\operatorname{inv}C(\mathbb{T})$. Since the map $z\mapsto\det((G_1(z))^*G_2(z))$ is continuous on $\overline{\mathbb{D}}$, it follows that the two maps $\zeta\mapsto\det((G_1(r\zeta))^*G_2(r\zeta))$ and $\zeta\mapsto\det((G_1(\zeta))^*G_2(\zeta))$ $(\zeta\in\mathbb{T})$ are close in the norm of $C(\mathbb{T})$ for all r's close enough to 1. Consequently their winding numbers are equal. Hence it follows that for a ρ sufficiently close to 1, when $\det(G_1^*G_2)$ is considered as an element F of $C_{\mathrm{b}}(\mathbb{A}_\rho)$, it is invertible in $C_{\mathrm{b}}(\mathbb{A}_\rho)$, we have that the F_r are invertible in $C(\mathbb{T})$ for all r's close enough to 1, and their winding numbers are 0. Thus the condition (C) is satisfied for

all ρ's close enough to 1. Hence $d_\nu^\rho(P_1,P_2) = \|\widetilde{G}_2 G_1\|_\infty = d_{\nu,\text{classical}}(P_1,P_2)$ for all ρ's close enough to 1. Consequently, $d_\nu^\infty(P_1,P_2) = d_{\nu,\text{classical}}(P_1,P_2)$ (<1).

Now suppose that $d_\nu^\infty(P_1,P_2) < 1$. Then $d_\nu^\rho(P_1,P_2)$ is a constant < 1 for all ρ's sufficiently close to 1. This implies that the condition (C) is satisfied for all ρ's close enough to 1. Hence the maps

$$\zeta \overset{\varphi_r}{\mapsto} \det((G_1(r\zeta))^* G_2(r\zeta)) \quad (\zeta \in \mathbb{T})$$

are all elements of inv $C(\mathbb{T})$ for all r's close enough to 1, and moreover, their winding numbers are all equal to 0. Owing to the invertibility in $C_{\text{b}}(\mathbb{A}_\rho)$, it follows that these maps φ_r are uniformly bounded away from 0 for all r's close enough to 1. Also, these maps converge in $C(\mathbb{T})$ to the map

$$\zeta \overset{\varphi}{\mapsto} \det((G_1(\zeta))^* G_2(\zeta)) \quad (\zeta \in \mathbb{T}).$$

Hence $\varphi \in \text{inv } C(\mathbb{T})$. Since the winding number map $w : \text{inv } C(\mathbb{T}) \to \mathbb{Z}$ is locally constant, we can also conclude that $w(\varphi) = 0$. Hence

$$d_{\nu,\text{classical}}(P_1,P_2) = \|\widetilde{G}_2 G_1\|_\infty = d_\nu^\infty(P_1,P_2) \ (<1).$$

This completes the proof. □

3.3. d_ν^∞ **is an extension of the ν-metric defined for $R = QA$ in [2].** In [2], a ν-metric was defined when $R = QA$, and we recall the definition below.

First of all, we use the notation QC for the C^*-subalgebra of $L^\infty(\mathbb{T})$ of *quasicontinuous* functions: $QC := (H^\infty + C(\mathbb{T})) \bigcap \overline{(H^\infty + C(\mathbb{T}))}$. The Banach algebra QA of analytic quasicontinuous functions is $QA := H^\infty \bigcap QC$. For $P_1, P_2 \in \mathbb{S}(QA,p,m)$, with the normalized left/right coprime factorizations

$$\begin{aligned} P_1 &= N_1 D_1^{-1} = \widetilde{D}_1^{-1}\widetilde{N}_1, \\ P_2 &= N_2 D_2^{-1} = \widetilde{D}_2^{-1}\widetilde{N}_2, \end{aligned}$$

we define

$$d_\nu(P_1,P_2) := \begin{cases} \|\widetilde{G}_2 G_1\|_\infty & \text{if } \det(G_1^* G_2) \in \text{inv } QC \text{ and} \\ & \qquad \text{Fredholm index of } T_{\det(G_1^* G_2)} = 0, \\ 1 & \text{otherwise.} \end{cases} \tag{3.7}$$

where the notation is as in Subsections 2.1-2.6.

In this subsection we will show that our ν-metric, defined by (3.5), coincides exactly with the above metric defined by (3.7), when the data P_1, P_2 belong to $\mathbb{S}(QA,p,m)$ (instead of the bigger set $\mathbb{S}(H^\infty,p,m)$).

If $\varphi \in L^1(\mathbb{T})$, then $\varphi_{(r)}$ $(0 \le r < 1)$ is the map defined by

$$\varphi_{(r)}(\zeta) = (f * P_r)(\zeta) \quad (\zeta \in \mathbb{T}).$$

Here P_r denotes the Poisson kernel, given by

$$P_r(\theta) = \sum_{k\in\mathbb{Z}} r^{|k|} e^{ik\theta} \quad \theta \in [0, 2\pi).$$

Then it is straightforward to see that $(\varphi^*)_{(r)} = (\varphi_{(r)})^*$. A result of Sarason [11, Lemma 6] says that for $\varphi \in QC$ and $\psi \in L^\infty(\mathbb{T})$,

$$\lim_{r\to 1} \|\varphi_{(r)}\psi_{(r)} - (\varphi\psi)_{(r)}\|_\infty = 0.$$

We will also use the result given below; see [4, Theorem 7.36], [9, Part B, Theorem 4.5.10].

Proposition 3.17. *If $f \in H^\infty + C(\mathbb{T})$, then T_f is Fredholm if and only if there exist $\delta, \epsilon > 0$ such that*

$$|f_{(r)}(e^{it})| \geq \epsilon \textit{ for } 1-\delta < r < 1,\ t \in [0, 2\pi).$$

Moreover, then the Fredholm index of T_f (namely, $\dim(\ker T_f) - \dim(\ker T_f^*)$) is the negative of the winding number with respect to the origin of the curves $f_{(r)}$ for $1-\delta < r < 1$.

Theorem 3.18. *Let $P_1, P_2 \in \mathbb{S}(QA, p, m)$. Then $d_{\nu,QA}(P_1, P_2) = d_\nu^\infty(P_1, P_2)$.*

Proof. Let $d_{\nu,QA}(P_1, P_2) < 1$. Then $\varphi := \det(G_1^* G_2) \in \text{inv } QC$. But then it is also invertible as an element of $H^\infty + C(\mathbb{T})$. From Douglas's result recalled above, we have that for all r sufficiently close to 1, $\varphi_{(r)} \in \text{inv } C(\mathbb{T})$, they are uniformly bounded away from 0, and their winding numbers are all equal to the Fredholm index of T_φ.

Using Sarason's result ([11, Lemma 6]) recalled above, and the local constancy of winding numbers, we will now show that for all r's close enough to 1 the maps $\zeta \overset{\varphi_r}{\mapsto} \det((G_1(r\zeta))^* G_2(r\zeta))$ ($\zeta \in \mathbb{T}$) are invertible as elements of $C(\mathbb{T})$, and moreover their winding numbers are all 0. Indeed, we have

$$\det((G_1|_{r\mathbb{T}})^* G_2|_{r\mathbb{T}}) = \sum_i (g_{1i}|_{r\mathbb{T}})^* (g_{2i}|_{r\mathbb{T}})$$

for suitable scalar $g_{1i}, g_{2i} \in QA$ and indices i. But by [9, Part A, Section 3.4], $g_{1i}|_{r\mathbb{T}} = g_{1i,(r)}$ and $g_{2i}|_{r\mathbb{T}} = g_{2i,(r)}$. Also, by Sarason's result, for all i's

$$\|g_{1i,(r)}^* g_{2i,(r)} - (g_{1i}^* g_{2i})_{(r)}\|_\infty \xrightarrow{r\to 1} 0.$$

Hence $\|\varphi_r - \varphi_{(r)}\|_\infty \xrightarrow{r\to 1} 0$. Since for all r's close enough to 1, the $\varphi_{(r)}$ are uniformly bounded away from 0, it follows that also the φ_r are uniformly bounded away from 0. In particular, they are all elements of inv $C(\mathbb{T})$ for r's sufficiently near 1. Finally, by the local constancy of winding numbers, it follows that also the winding numbers of φ_r are all 0 for all r's close enough to 1.

Hence when $\det(G_1^* G_2)$ is considered as an element F of $C_{\rm b}(\mathbb{A}_\rho)$, we have that F_r are invertible in $C(\mathbb{T})$ for all r's close enough to 1, and their winding numbers are 0. Thus the condition (C) is satisfied for all ρ's close enough

to 1. Hence $d_\nu^\rho(P_1,P_2) = \|\widetilde{G}_2 G_1\|_\infty = d_{\nu,QA}(P_1,P_2)$ for all ρ's close enough to 1. Consequently, $d_\nu^\infty(P_1,P_2) = d_{\nu,QA}(P_1,P_2)$ (< 1).

Now suppose that $d_\nu^\infty(P_1,P_2) < 1$. Then $d_\nu^\rho(P_1,P_2)$ is a constant < 1 for all ρ's sufficiently close to 1. This implies that the condition (C) is satisfied for all ρ's close enough to 1. Hence the maps

$$\zeta \stackrel{\varphi_r}{\mapsto} \det((G_1(r\zeta))^* G_2(r\zeta)) \quad (\zeta \in \mathbb{T})$$

are all elements of inv $C(\mathbb{T})$ for all r's close enough to 1, and moreover, their winding numbers are all equal to 0. Owing to the invertibility in $C_{\rm b}(\mathbb{A}_\rho)$, it follows that these maps φ_r are uniformly bounded away from 0 for all r's close enough to 1. Set φ to be the map

$$\zeta \stackrel{\varphi}{\mapsto} \det((G_1(\zeta))^* G_2(\zeta)) \quad (\zeta \in \mathbb{T})$$

From the above observations, the maps $\varphi_{(r)}$ are uniformly bounded away from 0 for all r's sufficiently near 1 and moreover their winding numbers are all 0. But then by Douglas's result recalled above (or see [9, Corollary 4.5.11]), the operator T_φ is invertible. In particular, it is Fredholm with Fredholm index 0. Hence $d_{\nu,QA}(P_1,P_2) = \|\widetilde{G}_2 G_1\|_\infty = d_\nu^\infty(P_1,P_2)$ (< 1). This completes the proof. □

4. A computational example

Consider the transfer function P given by

$$P(s) := e^{-sT}\frac{s}{s-a},$$

where $T, a > 0$. Thus $P \in \mathbb{F}(H^\infty(\mathbb{C}_{>0}))$, where $H^\infty(\mathbb{C}_{>0})$ denotes the set of bounded and holomorphic functions defined in the open right half plane $\mathbb{C}_{>0} := \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$. With the conformal map $\varphi : \mathbb{D} \to \mathbb{C}_{>0}$ given by

$$\varphi(z) = \frac{1+z}{1-z} \quad (z \in \mathbb{D}),$$

we can then transplant the plant to the unit disk. In this manner, we can endow $\mathbb{S}(H^\infty(\mathbb{C}_{>0}, p, m)$ also with the ν-metric. As an illustration, we will calculate the ν-metric between a pair of plants arising from this P when there is uncertainty in the parameter a or T. A normalized (left=right) coprime factorization of P above is given by $P = N/D$, where

$$N(s) = \frac{se^{-sT}}{\sqrt{2}s + a}, \quad D(s) = \frac{s-a}{\sqrt{2}s + a}.$$

4.1. Uncertainty in a.

Consider the two plants

$$P_1 := e^{-sT}\frac{s}{s-a_1} \text{ and } P_2 := e^{-sT}\frac{s}{s-a_2},$$

where $T, a_1, a_2 > 0$. Set $s := \varphi(z)$ for $z \in \mathbb{A}_\rho$, $\rho \in (0,1)$. Then

$$f(s) := G_1^* G_2 = \frac{|s|^2 e^{-2\mathrm{Re}(s)T} + (\overline{s} - a_1)(s - a_2)}{(\sqrt{2}\overline{s} + a_1)(\sqrt{2}s + a_2)} \quad (z \in \mathbb{A}_\rho).$$

It is clear that $z \mapsto |f(\varphi(z))|$ is bounded on $\mathbb{D}$. It can be shown that for $|a_1 - a_2|$ small enough, the real part of $f(s)$ is nonnegative and bounded away from zero for all $s \in \mathbb{C}$ such that $\mathrm{Re}(s) \geq 0$, as shown below.

Lemma 4.1. *Let $T, a > 0$. $\mathbb{C}_{\geq 0} := \{s \in \mathbb{C} : \mathrm{Re}(s) \geq 0\}$ and set*

$$f(s) := \frac{|s|^2 e^{-2\mathrm{Re}(s)T} + (\overline{s} - a)(s - a - \delta)}{(\sqrt{2}\overline{s} + a)(\sqrt{2}s + a + \delta)} \quad (s \in \mathbb{C}_{\geq 0}).$$

Then there is a δ_0 small enough such that for all $0 \leq \delta < \delta_0$, there is a $m > 0$ such that $\mathrm{Re}(f(s)) > m > 0$ $(s \in \mathbb{C}_{\geq 0})$.

Proof. Choose $\epsilon > 0$ such that $\dfrac{2\epsilon}{\sqrt{2}} + \epsilon^2 < \dfrac{1}{4}$. We have $\displaystyle\lim_{\substack{|s| \to \infty \\ s \in \mathbb{C}_{\geq 0}}} \frac{s - a}{\sqrt{2}s + a} = \frac{1}{\sqrt{2}}$. So we can choose a $R > 0$ such that $\left| \dfrac{s-a}{\sqrt{2}s + a} - \dfrac{1}{\sqrt{2}} \right| < \dfrac{\epsilon}{2}$. We have

$$\left| \frac{s-a}{\sqrt{2}s+a} - \frac{s-a-\delta}{\sqrt{2}s+a+\delta} \right| = (1+\sqrt{2})|\delta| \frac{|s|}{|\sqrt{2}s+a|} \frac{1}{|\sqrt{2}s+a+\delta|}.$$

It is easily seen that for all $s \in \mathbb{C}_{\geq 0}$, $\dfrac{|s|}{|\sqrt{2}s+a|} \leq \dfrac{1}{\sqrt{2}}$, and if $|s| \geq R$, then

$$\frac{1}{|\sqrt{2}s+a+\delta|} \leq \frac{1}{\sqrt{2}R}.$$

So we have that $\left| \dfrac{s-a}{\sqrt{2}s+a} - \dfrac{s-a-\delta}{\sqrt{2}s+a+\delta} \right| \leq \dfrac{1+\sqrt{2}}{2R} \cdot \delta$. Choose δ_0 so that

$$\frac{1+\sqrt{2}}{2R} \cdot \delta < \frac{\epsilon}{2}$$

for all $0 \leq \delta < \delta_0$. Thus whenever $|s| > R$, we have for all such δ that

$$\left| \frac{s-a-\delta}{\sqrt{2}s+a+\delta} - \frac{1}{\sqrt{2}} \right| < \epsilon.$$

Hence $\left| \dfrac{s-a}{\sqrt{2}s+a} \cdot \dfrac{s-a-\delta}{\sqrt{2}s+a+\delta} - \dfrac{1}{2} \right| < \dfrac{2\epsilon}{\sqrt{2}} + \epsilon^2 < \dfrac{1}{4}$. Thus

$$\frac{1}{2} - \mathrm{Re}\left(\frac{s-a}{\sqrt{2}s+a} \cdot \frac{s-a-\delta}{\sqrt{2}s+a+\delta} \right) \leq \left| \frac{s-a}{\sqrt{2}s+a} \cdot \frac{s-a-\delta}{\sqrt{2}s+a+\delta} - \frac{1}{2} \right| < \frac{1}{4},$$

and so $\mathrm{Re}\left(\dfrac{s-a}{\sqrt{2}s+a} \cdot \dfrac{s-a-\delta}{\sqrt{2}s+a+\delta} \right) > \dfrac{1}{4}$. But clearly for $s \in \mathbb{C}_{\geq 0}$,

$$\mathrm{Re}\left(\frac{|s|^2 e^{-2\mathrm{Re}(s)T}}{(\sqrt{2}\overline{s}+a)(\sqrt{2}\overline{s}+a+\delta)} \right) \geq 0.$$

Hence $\mathrm{Re}(f(s)) \geq \dfrac{1}{4}$ for $|s| > R$ and $0 \leq \delta < \delta_0$.

Set $K := \{s \in \mathbb{C} : |s| \le R\} \bigcap \mathbb{C}_{\ge 0}$. Define $F : K \times [0,1] \to \mathbb{R}$ by

$$F(s,\delta) = \operatorname{Re}\left(\frac{|s|^2 e^{-2\operatorname{Re}(s)T} + (\overline{s}-a)(s-a-\delta)}{(\sqrt{2}\overline{s}+a)(\sqrt{2}s+a+\delta)}\right) \quad (s \in K,\ \delta \in [0,1]).$$

Then $F(s,0) = \operatorname{Re}\left(\dfrac{|s|^2 e^{-2\operatorname{Re}(s)T} + |s-a|^2}{|\sqrt{2}\overline{s}+a|^2}\right) \ge 0$. Set $2m := \min\limits_{s\in K} F(s,0)$. Clearly $m \ge 0$. In fact, $m > 0$ since if $F(s_0, 0) = 2m = 0$ for some $s_0 \in K$, then we would have $|s_0 - a|^2 = 0$, and so $s_0 = a$, but then

$$2m = |s|^2 e^{-2\operatorname{Re}(s)T}|_{s=s_0=a} \neq 0,$$

a contradiction. As F is continuous on the compact set $K \times [0,1]$, it is uniformly continuous there. Refine the choice of δ_0 if necessary so that $0 \le \delta < \delta_0$ implies that $|F(s,\delta) - F(s,0)| < m$ for all $s \in K$. Hence we have that $F(s,\delta) = \operatorname{Re}(f(s)) > m$ for all $0 \le \delta < \delta_0$ and $s \in K$. This completes the proof. ☐

In light of the above result, $G_1^* G_2 \in \operatorname{inv} C_{\mathrm{b}}(\mathbb{A}_\rho)$ for ρ close enough to 1, and $W(G_1^* G_2) = 0$. We also have

$$\widetilde{G}_2 G_1 = \frac{s e^{-sT}(a_2 - a_1)}{(\sqrt{2}s + a_1)(\sqrt{2}s + a_2)},$$

where $s := \varphi(z)$, $z \in \mathbb{T} \setminus \{1\}$. Consequently, using the Cauchy-Schwarz (in)equality, we obtain

$$\|\widetilde{G}_2 G_1\|_\infty = \frac{|a_2 - a_1|}{2} \sup_{\omega \ge 0} \frac{\omega}{\sqrt{\omega^2 + \frac{a_1^2}{2}}\sqrt{\omega^2 + \frac{a_2^2}{2}}} = \frac{|a_2 - a_1|}{2} \frac{\sqrt{2}}{a_1 + a_2}.$$

Hence

$$d_\nu^\infty(P_1, P_2) = \frac{|a_1 - a_2|}{\sqrt{2}(a_1 + a_2)}$$

whenever $|a_1 - a_2|$ is small enough.

4.2. **Uncertainty in T.** Consider the two plants

$$P_1 := e^{-sT_1} \frac{s}{s-a} \text{ and } P_2 := e^{-sT_2} \frac{s}{s-a},$$

where $T_1, T_2, a > 0$. We will show that $\|\widetilde{G}_2 G_1\|_\infty = 1$, and so irrespective of whether or not the condition (C) is satisfied for some ρ, the ν-metric between the plants will be always 1.

We have

$$\widetilde{G}_2 G_1 = \frac{s(s-1)(e^{-sT_2} - e^{-sT_1})}{2\left(s + \frac{1}{\sqrt{2}}\right)^2},$$

where $s := \varphi(z)$, $z \in \mathbb{T} \setminus \{1\}$. Thus

$$\|\widetilde{G}_2 G_1\|_\infty = \sup_{\omega \geq 0} \frac{\omega\sqrt{\omega^2+1}}{2\sqrt{\omega^2+\frac{1}{2}}}\sqrt{2}\sqrt{1-\cos(\omega(T_2-T_1))}.$$

By the Arithmetic Mean-Geometric Mean inequality, we have for $\omega \geq 0$ that

$$\omega^2(\omega^2+1) \leq \left(\frac{\omega^2+(\omega^2+1)}{2}\right)^2 = \left(\omega^2+\frac{1}{2}\right)^2.$$

We have

$$\sup_{\omega\geq 0} \frac{\omega\sqrt{\omega^2+1}}{\sqrt{\omega^2+\frac{1}{2}}} = 1 = \lim_{\omega\to\infty} \frac{\omega\sqrt{\omega^2+1}}{\sqrt{\omega^2+\frac{1}{2}}}.$$

Also with

$$\omega := \frac{(2n+1)\pi}{T_2-T_1} \quad (n \in \mathbb{N})$$

we have that $\omega \to \infty$, and $\cos(\omega(T_2-T_1)) = -1$. Thus $\|\widetilde{G}_2 G_1\|_\infty = 1$, and so

$$d_\nu^\infty(P_1, P_2) = 1.$$

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